MEM6810 Engineering Systems Modeling and Simulation

Sino-US Global Logistics Institute
Shanghai Jiao Tong University

Spring 2024 (full-time)

Assignment 1

Due Date: April 2 (in class)

Instruction

(a) You can answer in English or Chinese or both.
(b) Show **enough** intermediate steps.
(c) Write your answers **independently**.
(d) If you copy the solutions from somewhere, you must **indicate the source**.

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Question 1 ($6 \times 5 = 30$ points)

Are the following statements true or false for general case? If true, prove it; otherwise, give a counter-example.

(1) If X is independent of Y, Y is independent of Z, then X is independent of Z.
(2) X and Y are two random variables. If X^2 is independent of Y^2, then X is independent of Y.
(3) X and Y are two independent random variables. Let $g(x)$ be a function only of x and $h(y)$ be a function only of y. Then, $g(X)$ is independent of $h(Y)$.
(4) If $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$, then X is independent of Y.
(5) $\rho(X, X^2)$ must be nonzero.

Question 2 (10 points)

$Y \sim \text{geometric}(p)$, prove that $\mathbb{E}[X] = 1/p$ and $\text{Var}(X) = (1-p)/p^2$. (Do not directly use the property of negative binomial distribution; use the definition.)

Question 3 (10 points)

If $X, Y \sim \text{Exp}(\lambda)$ and they are independent, prove that $X + Y \sim \text{Erlang}(2, \lambda)$. (Do not directly use the property of Erlang distribution or Gamma distribution; use the definition.)

Question 4 (10 points)

If $X \sim \text{Exp}(\lambda)$, prove that $cX \sim \text{Exp}(\lambda/c)$ for $c > 0$. (Do not directly use the property of Erlang distribution or Gamma distribution; use the definition.)

Question 5 (5 points)

If $X_1, X_2, \dots, X_n$ are n independent random variables, and $X_i \sim \text{Exp}(\lambda_i)$, $i = 1, \dots, n$, prove that

$$\min\{X_1, \dots, X_n\} \sim \text{Exp}(\lambda_1 + \cdots + \lambda_n).$$

Question 6 ($5 \times 3 = 15$ points)

Recall the definition of a.s. convergence:

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

(1) Is the above definition of a.s.convergence equivalent to the following definition?

$$\mathbb{P}\left\{\omega \in \Omega : \forall \epsilon > 0,\ \exists N_{\epsilon,\omega} \text{ s.t. } \forall n \geq N_{\epsilon,\omega},\ |X_n(\omega) - X(\omega)| \leq \epsilon\right\} = 1,$$

where $\forall$ means "for any", $\exists$ means "there exists", and s.t. means "such that".

(2) Prove that, for any fixed ω, the following two things are equivalent:

$$\forall \epsilon > 0,\ \exists N_{\epsilon,\omega} \text{ s.t. } \forall n \geq N_{\epsilon,\omega},\ |X_n(\omega) - X(\omega)| \leq \epsilon,$$

$$\forall m \in \mathbb{N}^+,\ \exists N_{m,\omega} \text{ s.t. } \forall n \geq N_{m,\omega},\ |X_n(\omega) - X(\omega)| \leq 1/m,$$

where $\mathbb{N}^+$ denotes the set of natural numbers.

(3) Prove that, if $\mathbb{P}(|X_n - X| > \epsilon \text{ i.o.}) = 0$ for any $\epsilon > 0$, then $X_n \xrightarrow{a.s.} X$. [Hint: De Morgan's laws: Consider sets $A_i,\ i \in I$, where I can be either a countable set or an uncountable set. Let $\overline{A}$ denote the complement of a set A. Then $\overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$, and $\overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \overline{A_i}$. Boole's inequality: When I is a countable set, $\mathbb{P}(\bigcup_{i\in I} A_i) \leq \sum_{i\in I} \mathbb{P}(A_i)$.]

Question 7 (10 points)

Recall the Numerical Integration example in Lec 1 page 28/33. Suppose that $f(x)$ is continuous on $[a, b]$. Let

$$Y_n := \frac{b-a}{n}[f(X_1) + \cdots + f(X_n)].$$

Prove that $Y_n \xrightarrow{a.s.} \int_a^b f(x)\mathrm{d}x$ as $n \to \infty$.

Question 8 (10 points)

Prove that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ in the normal distribution case. [Hint: Let S_n^2 denote the sample variance when the sample size is n. First show that $\frac{S_2^2}{\sigma^2} \sim \chi^2_1$. Then reach the conclusion for $\frac{(n-1)S_n^2}{\sigma^2}$ by induction.